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Bessel functions of two variables: some power series and plots

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Abstract

For a finite reflection group G there is a rich theory developed by Dunkl, Heckman and Opdam leading to the notion of a commuting set of Bessel differential operators. These systems play an important role in the study of Calogero–Moser systems and other problems of physical interest. When G acts on the line one recovers the usual Bessel function with a well known power series expansion at the origin. We obtain some such expansions in the case of $G = A_2$ acting in the plane and we use these to produce plots of some of these functions.

(Some figures in this article are in colour only in the electronic version; see `www.iop.org`)

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Introduction

Special functions such as Gauss' hypergeometric series and the Bessel functions have played a crucial role in mathematical physics for a very long time, and many branches of engineering and technology provide good examples of such use. The appearance of symbolic/numerical/graphics packages such as Macsyma, Maple, Mathematica and others has put plots of these functions at the fingertips of many users in applied fields.

In the last ten years, the groundbreaking work of Dunkl, Heckman and Opdam, de Jeu and others has produced a rich theory of Bessel functions of several variables which extends in a natural way the one-dimensional situation. There are also several versions of Gauss' function in the case of several variables, and a large program dealing with polynomials has been developed by Macdonald, Koornwinder and Cherednik, as well as others. These functions have arisen at times in connection with integrable systems or as spherical functions for appropriate symmetric spaces. One can wonder if these relatively recent mathematical objects will eventually have an impact on technology comparable to the one-variable case, and how much they will penetrate the engineering literature.

There are at least three interrelated (and rather long-winded) answers that I can think of:

(a) This will happen if interesting 'down-to-earth' problems ever get solved in terms of these new functions. In the one-dimensional case these applications are very well known in mathematical physics, but they reach into many other areas. We just point out that in probability theory they give the (non-trivial part of the) natural analogue of the Gaussian kernel when the real line is replaced by the integers. For this and many more uses of Bessel functions, see [F].

(b) The amount of heavy-going mathematical sophistication needed to go through this new material makes it unlikely that enough 'applied math' people will take the time to learn about these new tools and test their applicability. Whereas in the case of one variable the theory developed after (or hand-in-hand with) the applications, in the general case the only applications that come to mind are the Calogero–Moser models of interacting particles.

(c) Maybe a few plots will not hurt. After all the best way to see the usefulness of sines and cosines—as well as Bessel functions, elliptic functions, and many other such gems—is to view a few graphs of them. Keep in mind that whether we like it or not, more and more engineering students will be trained in front of a screen with increasing graphical capabilities.

The goal of this paper is rather modest. I want to show some of these plots and hopefully provoke someone into doing a better job. Given the quality of the graphs that I show, this should not be too much of a challenge. I have not seen any such graphs in the literature or even some of the power series expressions that I will use to compute with. It is clear that for computational purposes eventually one would like something smarter than power series, like piecewise rational approximation or similar things. This can wait until we see whether these new functions get used enough so as to warrant such an effort.

The eigenvalue problem

On the plane with coordinates (a, b) consider the operators L_1 and L_2 given by

$$L_1(f) = \frac{\mathrm{d}^2 f}{\mathrm{d}b^2} + \frac{\mathrm{d}^2 f}{\mathrm{d}a^2} + k\left(\frac{1}{(\sqrt{3}b+3a)^2} + \frac{1}{(\sqrt{3}b-3a)^2} + \frac{1}{12b^2}\right) f$$

$$L_2(f) = \frac{27k}{2\sqrt{3}}\left(\frac{1}{(\sqrt{3}b-3a)^2} - \frac{1}{(\sqrt{3}b+3a)^2}\right)\frac{\mathrm{d}f}{\mathrm{d}b} + \frac{\mathrm{d}^3 f}{\mathrm{d}a^3} - 3\frac{\mathrm{d}^3 f}{\mathrm{d}a\mathrm{d}b^2} + \frac{3k}{2}\left(\frac{1}{(\sqrt{3}b+3a)^2} + \frac{1}{(\sqrt{3}b-3a)^2} - \frac{1}{6b^2}\right)\frac{\mathrm{d}f}{\mathrm{d}a}.$$

Here k is an arbitrary parameter, and it is related to the one in [O1], denoted here by k_0, by $k = 3(1 - k_0)k_0$.

These operators are invariant under the operations (1) b goes into $-b$, as well as (2) (a, b) goes into $(-a/2 + (\sqrt{3}/2)b, (\sqrt{3}/2)a + b/2)$. The first operation is a reflection across the a axis, the second one a reflection across the axis given by the vector $(-\sqrt{3}, 1)$. The same is true if one considers one more reflection, across the axis making an angle of $-60°$ with the a axis. There is then a three-fold symmetry, and since these reflections generate a group of six elements (the symmetric group on three symbols) the full group of symmetries has order six. The operators L_1, L_2 correspond to the fundamental invariants for the group A_2 in question.

The operators commute and one can define the Bessel function as the function made symmetric by adding six terms of the form

$$f(a, b, s_1, s_2) = \mathrm{e}^{s_1 a + s_2 b}(1 + \text{small at infinity})$$

that solves the system

$$L_1(f) = \lambda_1 f \qquad \text{and} \qquad L_2(f) = \lambda_2 f.$$

This clearly requires $\lambda_1 = s_1^2 + s_2^2$ and $\lambda_2 = s_1^3 - 3s_1 s_2^2$.

A very careful referee pointed out that this asymptotic form of the Bessel function is, in general, still conjectural. The relation connecting (λ_1, λ_2) and (s_1, s_2) depends on this very natural conjecture.

Put $a = r\cos t$ and $b = r\sin t$ and put $x = \cos 3t$ and make the analogous change of variables on the spectral side $s_1 = \beta\cos s$, $s_2 = \beta\sin s$, $y = \cos 3s$. For later use, introduce p as any root of the equation

$$k = -\tfrac{1}{3}(4p^2 - 12p).$$

In terms of the parameter k_0 in [O1] we have

$$p = \tfrac{3}{2}(1 \pm \sqrt{1 + k_0(k_0 - 1)}) \qquad k_0 = \tfrac{1}{2} \pm \tfrac{1}{6}\sqrt{16p(p-3) + 9}.$$

Conjugate the resulting operators by the factor

$$r^p(1 - x^2)^{p/6}$$

i.e. define $\mathcal{L}_i$ by

$$r^p(1 - x^2)^{p/6}\,\mathcal{L}_i = L_i\, r^p(1 - x^2)^{p/6}$$

to get new operators $\mathcal{L}_i$ $(i = 1, 2)$ given by

$$\mathcal{L}_1 f = -\frac{9(x-1)(x+1)}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \frac{3(2p+3)x}{r^2}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{(2p+1)}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}$$

and

$$\begin{aligned}\mathcal{L}_2 f = &-\frac{3(35x^2 - 4p^2 + 6p - 17)}{r^3}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{9(6x^2 + 2p - 3)}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r\mathrm{d}x}\\ &-\frac{27(x-1)^2(x+1)^2}{r^3}\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} + \frac{27(x-1)x(x+1)}{r^2}\frac{\mathrm{d}^3 f}{\mathrm{d}r\mathrm{d}x^2}\\ &-\frac{135(x-1)x(x+1)}{r^3}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \frac{9(x-1)(x+1)}{r}\frac{\mathrm{d}^3 f}{\mathrm{d}r^2\mathrm{d}x}\\ &-\frac{3x}{r}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{3x}{r^2}\frac{\mathrm{d}f}{\mathrm{d}r} + x\frac{\mathrm{d}^3 f}{\mathrm{d}r^3}.\end{aligned}$$

Consider now the resulting eigenvalue problem

$$\begin{aligned}\mathcal{L}_1\, f &= \beta^2 f\\ \mathcal{L}_2\, f &= \beta^3 y f.\end{aligned}$$

It is easy to see that this admits solutions of the form

$$1 + \sum c_{ijkl} r^{2i}(r^3 x)^j \beta^{2k}(\beta^3 y)^l$$

with $c_{ijkl} = c_{klij}$.

The plots in figure 1 are produced from this power series which is completely consistent with the symmetrized asymptotic form discussed earlier.

In particular we get the well known fact that the Bessel function is symmetric in the interchange between 'spatial' and 'spectral' variables. This is then a 'trivial' or 'basic' instance of the bispectral property discussed in [DG] but for systems of partial differential operators. In this connection it is pleasing to see that the relations

$$ad^3(\mathcal{L}_i)(r^2) = 0 \qquad i = 1, 2$$

and

$$ad^4(\mathcal{L}_i)(r^3 x) = 0 \qquad i = 1, 2$$

hold in this case. Here r^2 and $r^3 x$ act as multiplication operators as in [DG].

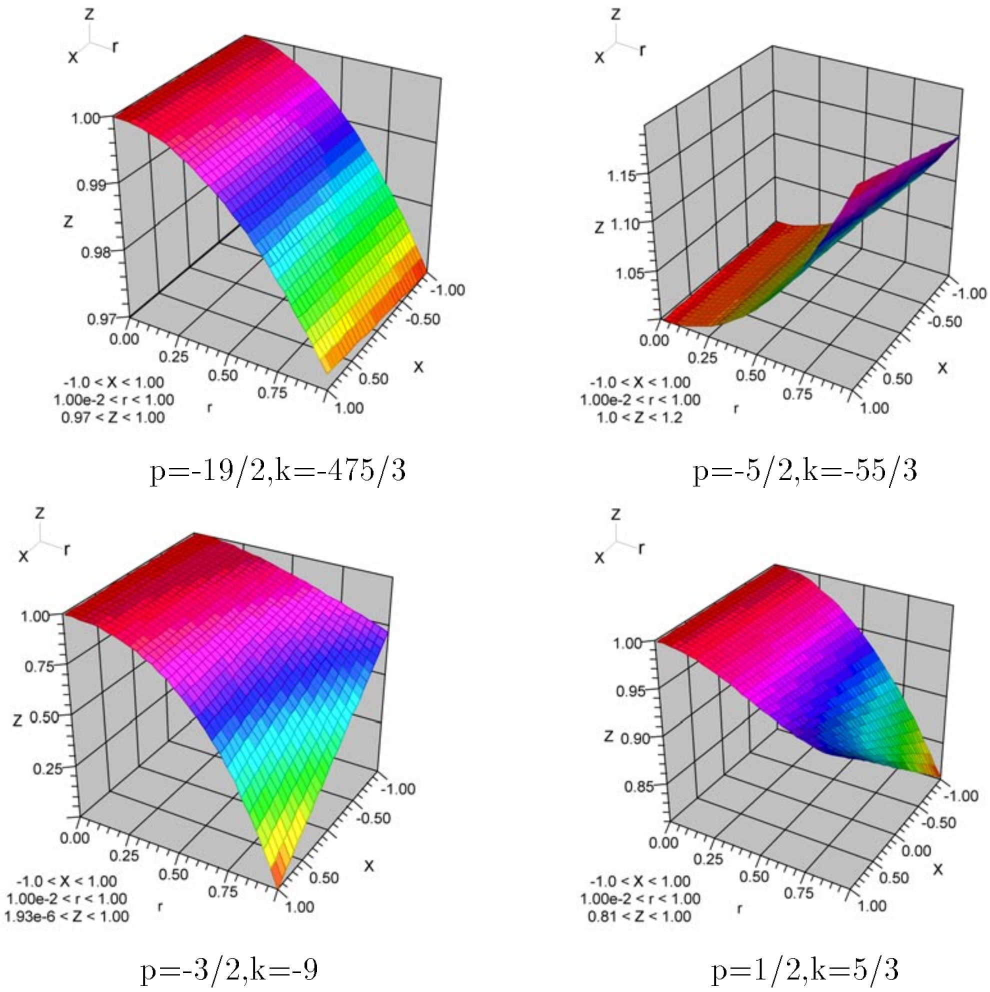


Figure 1. Functions of x, r given by the series $1 + \sum c_{ijkl} r^{2i} (r^3 x)^j \beta^{2k} (\beta^3 y)^l$ as well as their product with the conjugating factor $r^p (1 - x^2)^{p/6}$. The first four plots do not include this factor, and the last six do.

If we are in the simpler case when $\beta = y = 1$ then we have solutions of the form

$$1 + \sum c_{ij} r^{2i+3j} x^j$$

to the equations

$$\mathcal{L}_1 f = f \qquad \mathcal{L}_2 f = f.$$

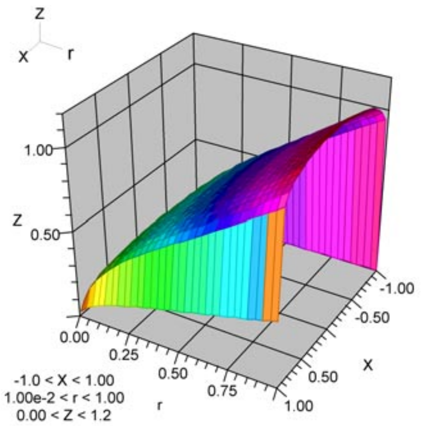
For this case we have found a simple expression for the coefficients c_{ij}, namely

$$c_{ij} = \frac{\Gamma(p)}{\Gamma(p/3)} \frac{3^i \Gamma(p/3 + i + j)}{2^{2j+2i} i!\, j!\, \Gamma(p + 2i + 3j)}$$

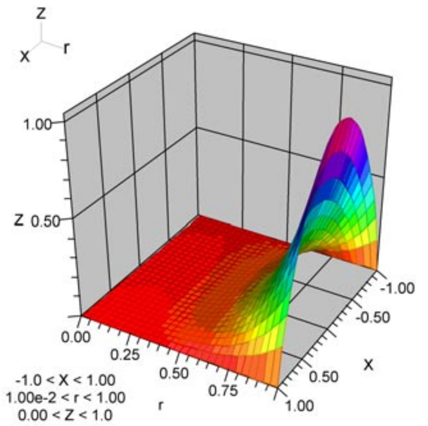
and this will be used in the plots of figure 1. These plots, made for different values of p, give an indication of f for $x = \cos 3\theta$ in $(-1, 1)$ and r in $(0, 1)$.

To get closer to the true Bessel functions we still need to multiply these expressions by the conjugating factor

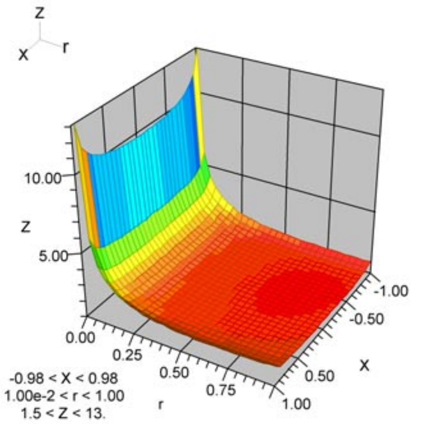
$$r^p (1 - x^2)^{p/6}$$



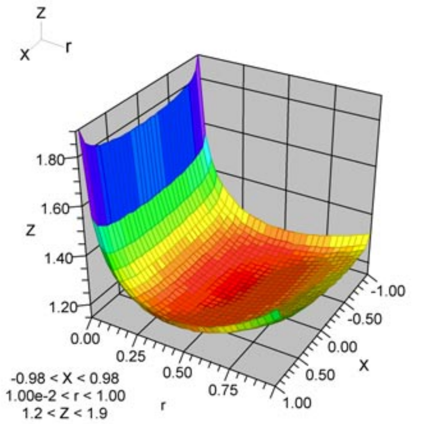
p=1/2,k=5/3



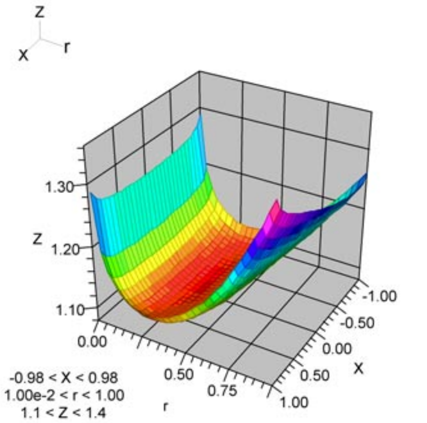
p=5,k=-40/3



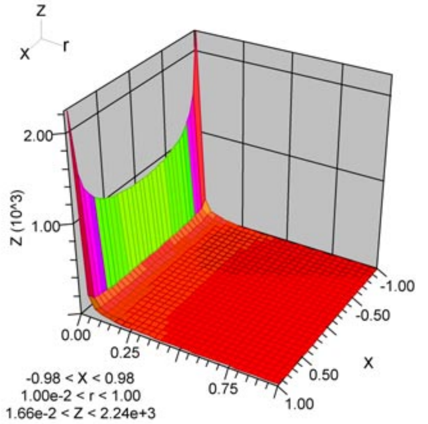
p=-1/2,k=-7/3



p=-1/8,k=-25/48



p=-1/20,k=-61/300



p=-3/2,k=-9

Figure 1. (Continued)

used above. To bring this into full agreement with the definition in [O1] one still needs to divide by the function I introduced in page 341 of [O1]. This would bring in an extra factor which is a power of $r^6(1 - x^2)$.

For convenience we plot both the function of x, r given by the series above as well as their product with this conjugating factor. The first four plots do not include this factor, and the last six do.

E Opdam [O2] has mentioned to me that M de Jeu has made some relevant computations, and I have received some notes from de Jeu [dJ2] who has obtained other expressions which could be used in producing plots of the Bessel functions. There are also expressions for those Bessel functions in [O] that could be used for plotting purposes.

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